

# The Area Spectrum in Quantum Gravity

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## Abstract

We show that, apart from the usual area operator of non-perturbative quantum gravity, there exists another, closely related, operator that measures areas of surfaces. Both corresponding classical expressions yield the area. Quantum mechanically, however, the spectra of the two operators are different, coinciding only in the limit when the spins labelling the state are large. We argue that both operators are legitimate quantum operators, and which one to use depends on the context of a physical problem of interest. Thus, for example, we argue that it is the operator proposed here that is relevant to use in the black hole context as measuring the area of black hole horizon. We show that the difference between the two operators is due to non-commutativity that is known to arise in the quantum theory. We give a heuristic picture explaining the difference between the two area spectra in terms of quantum fluctuations of the surface whose area is being measured.

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A well-known result of the loop approach [1] to quantum gravity is that the area of surfaces is quantized. In this theory the spectrum of the operator measuring the area of a surface  $S$  is given by [2]

$$A_S = 8\pi\gamma\ell_P^2 \sum_p \sqrt{j_p(j_p + 1)}, \quad (1)$$

where the sum is taken over all points  $p$  on the surface  $S$  where edges of a spin network state transversely intersect  $S$ ,  $j_p$  are spins (half-integers) that label the intersecting edges,  $\gamma$  is a real, positive parameter, known as Immirzi parameter [3], and  $\ell_P^2 = G\hbar$ ,  $G$  being Newton constant. It is assumed in (1) that all intersections are transversal.

In this note we show that another quantity, that in the classical theory coincides with the area of  $S$ , can be constructed, but which gives rise to a quantum operator with a *different* spectrum. We shall refer to this other area operator and to its eigenvalues as  $\tilde{A}_S$ . As we show below, the eigenvalues  $\tilde{A}_S$  are given by

$$\tilde{A}_S = 8\pi\gamma\ell_P^2 \sum_p j_p, \quad (2)$$

where the meaning of all the symbols is the same as in (1). Thus,  $\tilde{A}_S$  coincides with  $A_S$  only in the limit all spins  $j_p$  are large. We shall argue that both  $A_S$  and  $\tilde{A}_S$  give rise to ‘legitimate’ area operators, and a choice which of the area operators should be used depends on the context of a physical problem of interest. We also give a heuristic interpretation of the area spectra (1), (2) explaining the discrepancy between the two. The area operator  $\tilde{A}_S$  was also studied by Baez [4], in a similar context.

Before we give a description of the operator  $\tilde{A}_S$ , let us recall how the usual area operator [2] is constructed. Our choice of conventions is given in the Appendix. The area of a 2-surface  $S$  is given by

$$A_S = \int_S dx^1 \wedge dx^2 [\text{Tr}(\tilde{E}^3 \tilde{E}^3)]^{1/2}, \quad (3)$$

where we have used adapted coordinates such that  $S$  is given by  $x^3 = 0$ . A simple way to understand the area spectrum (1) is to rewrite the area operator as a sum of angular momentum operators. In quantum theory the quantity  $\tilde{E}^{3i}$  gets replaced by the operator of variational derivative  $i\hbar 8\pi G\gamma(\delta/\delta A_{3i})$ . The operator  $i(\delta/\delta A_{3i})$ , when acting on the holonomy along a spin net edge intersecting  $S$ , behaves as the product of the 2-dimensional delta-function and the angular momentum operator acting on the copy of the gauge group corresponding to the edge. Introducing the angular momentum operators  $X_p^i$  (one for each point  $p$  where  $S$  is intersected with an edge of a spin network), one can rewrite the area operator in the following simple form

$$\hat{A}_S = 8\pi\ell_P^2\gamma \sum_p [X_p^i X_{pi}^i]^{1/2}, \quad (4)$$

where the angular momentum operators satisfy the commutation relations of (times 1/2) Pauli matrices

$$[X_p^i, X_p^j] = i\varepsilon^{ijk} X_p^k. \quad (5)$$

Thus, each  $X_p^i$  is the usual angular momentum operator of quantum mechanics, which explains the spectrum (1). For details of this construction see [2].

The expression (3) giving the area of  $S$  is a (gauge-invariant) functional on the phase space of the theory. The other area operator we are going to construct is *not* a quantization of a gauge-invariant functional on the phase space. To construct it we work in the space of non gauge-invariant states. Let us fix a unit vector  $r^i$  (with values in the Lie algebra of  $SU(2)$ ) defined at each point on  $S$ . For an arbitrary unit vector  $r^i : r^i r_i = 1$  let us consider the following quantity:

$$\int_S dx^1 \wedge dx^2 \tilde{E}^{3i} r_i. \quad (6)$$

So far the quantity constructed has little to do with the area of  $S$ . Indeed, it is not gauge-invariant and not positive definite. However, one can extremize the quantity (6) by performing a gauge rotation on the  $\tilde{E}^a$  field and making it point in the same direction (in the internal space) as  $r^i$  at each point on  $S$ . Clearly, (6) extremized in this way is just the area of  $S$ . By  $\hat{A}_S$  we will always mean the quantity (6) where  $\tilde{E}^{3i}$  is chosen to point in the same direction as  $r^i$ .

Although the above procedure gives the area of  $S$  on the classical level, it may seem to be very hard to perform the ‘extremization’ procedure in the quantum theory. We shall see, however, that there is a natural analog of this ‘extremization’. It is not hard to construct an operator corresponding to (6). Indeed, one proceeds along the lines of the construction of the operator  $\hat{A}_S$ , replacing  $i(\delta/\delta A_{3i})$  by the two-dimensional delta-function times the angular momentum operator  $X_p^i$ . The corresponding operator is given by

$$8\pi\ell_P^2\gamma \sum_p X_p^i r_i. \quad (7)$$

Note that this operator is defined only in the space of non gauge-invariant states. It is easy to find its eigenvalues. Denoting the eigenvalues of the  $X_p^i r_i$  operator by  $m$  ( $m$  are half-integers), we get for the eigenvalues of the operator corresponding to (6)

$$8\pi\ell_P^2\gamma \sum_p m_p, \quad (8)$$

where the eigenvalues  $m_p : |m_p| \leq j_p$ ,  $j_p$  being the spins labelling the edges. Note that spin networks are *not* eigenstates of (7).

It is now easy to see what we have to do to recover the area operator from (7). Different eigenvalues (8) have natural interpretation as different ways to project  $\tilde{E}^{3i}$  vector on  $r^i$  direction. Classically one gets the area of  $S$  when  $\tilde{E}^{3i}$  points in the same direction as  $r^i$  and this is the maximal value for (6) one can get. The quantum mechanical analog of this is to take the maximal eigenvalue of the operator corresponding to (6). It is natural to interpret this maximal eigenvalue as the eigenvalue of the area operator  $\hat{A}_S$

$$\tilde{A}_S = 8\pi\ell_P^2\gamma \sum_p j_p. \quad (9)$$

Thus, as claimed, one can indeed construct an expression for the area different from (3). Classically, both expression give the area of  $S$ , quantum mechanically, however, the spectra of the corresponding operators are different, coinciding only in the limit of large spins.

It may seem that the area operator  $\tilde{A}_S$  is pathological. Indeed, it is defined using a quantity that is not gauge-invariant, and the construction involves the procedure of extremization, i.e., of taking the maximal eigenvalue, which may not be easy to make precise in the quantum theory. However, there is one context in which the area operator  $\tilde{A}_S$  is certainly legitimate. It is the case of quantum theory on a manifold with boundary. Dealing with such a theory one usually has to impose boundary conditions appropriate to the physical problem of interest. These boundary conditions may be such that they partially or completely brake gauge-invariance on the boundary. For example, in the construction of the black hole sector of quantum theory in [6] a unit vector  $r^i$  has to be fixed on one of the boundaries of space (black hole horizon), and the only gauge transformations that survive on the boundary are the ones fixing  $r^i$ . If this is the case, the quantity (6) constructed on the boundary is a gauge-invariant functional on the phase space. The area  $\tilde{A}_S$  is a legitimate area operator in this context, and its eigenvalues are given by (9).

In our opinion, both area operators discussed are of physical interest, and which operator one has to use depends on the context of a physical problem one deals with. Thus, it would be good to have an interpretation of the two area operators and to understand better the discrepancy in their spectra.

We have given expressions (4), (7) for the two operators in terms of angular momentum operators  $X_p^i$ . Both operators have a similar structure: they are both given by the sum over the intersection points  $p$ . Thus, to understand the difference between them, we can concentrate on what happens at a single intersection point. The contribution from each point  $p$  to (4) is just the square root of the operator  $(X^i X_i) \equiv X^2$ , the contribution to (7) is the projection of  $X^i$  on the direction  $r^i$ , where  $X^i$  is the usual angular momentum operator. Thus, as we know from the theory of angular momentum, the difference between the spectra (1), (2) arises because the components of the angular momentum do not commute (5). This is why the eigenvalue of  $X^2$  is given by  $j(j+1)$ , not by  $j^2$ . One can understand the discrepancy by using a heuristic idea of uncertainty that arises because of the non-commutativity of the components of  $X^i$ . Let us consider a representation of highest weight  $j$ , and let us visualize the angular momentum in this representation as a vector in  $\mathbb{R}^3$ . The length of this vector is  $\sqrt{j(j+1)}$ . The vector  $r^i$  gives a preferred direction in this internal space, and the maximal eigenvalue of  $(X \cdot r)$  operator in this representation is  $j$ . This can be visualized as the maximal projection of  $X^i$  vector on  $r^i$  direction allowed by the uncertainty relations. Indeed, the vector  $X^i$  can not point solely in the direction  $r^i$  because this would mean that the other (orthogonal to  $r^i$ ) components of the vector are zero. However, if at least one component of  $X^i$  is non-zero, all three components of the vector  $X^i$  cannot have definite values simultaneously, due to their non-commutativity. A maximal possible value of the projection of  $X^i$  on  $r^i$  direction must be accompanied by non-zero (and undetermined) values of the other two components of  $X^i$ , such that  $(X^\perp)^2 := X^2 - (X \cdot r)^2$  is equal to  $j$ . This can be visualized as vector  $X^i$  precessing (fluctuating) about  $r^i$  direction, with neither of the two components of  $X^\perp$  having definite values, but satisfying  $(X^\perp)^2 = j$ .

Thus, we get the following interpretation of the discrepancy between the two area operators. At each intersection point  $p$  of the surface  $S$  with an edge of a spin network, one has the quantum angular momentum vector  $X_p^i$ . The area operator (4) measures the length  $\sqrt{j_p(j_p+1)}$  of this vector, while the other area operator (7) measures the maximal possible

projection  $j_p$  of  $X_p^i$  on  $r^i$  direction. The two differ because of the non-commutativity (and associated uncertainty) of the components of vector  $X_p^i$ . Note that the non-commutativity we have discussed is closely related to the non-commutativity of area operators discussed in [5].

This picture can be further visualized by choosing an identification of the internal space  $\mathbb{R}^3$  at each point of  $S$  with the tangent space to the manifold. This is equivalent to choosing some (non-physical) triad field. Let us choose an identification in such a way that 2-flats orthogonal to the image of  $r^i$  are integrable and span the surface  $S$ . Thus, we will visualize  $r^i$  as the vector normal to the surface  $S$ . Then  $r^i(X_p \cdot r)$  is a vector at  $p$  orthogonal to  $S$ , whose length is  $j_p$ . We can associate with it a 2-flat at  $p$ , with the area  $8\pi\ell_P^2\gamma j_p$ . Then the area (2) of  $S$  is simply the sum of contributions coming from all such 2-flats. The vector  $X_p^i$  has the length  $\sqrt{j_p(j_p + 1)}$ , and we can associate with it a 2-flat at  $p$  of the area  $8\pi\ell_P^2\gamma\sqrt{j_p(j_p + 1)}$ . However, in the visual picture this vector (and the corresponding two-flat) is constantly fluctuating. The collection of 2-flats corresponding to  $X_p^i$  does no longer span the surface  $S$ , but can be visualized as  $S$  fluctuating. The area of this fluctuating surface is given by the sum of the areas of the 2-flats, i.e., by (1).

Thus, the interpretation of the two areas (1), (2) we propose is that quantum mechanically there are two surfaces instead of one classical, and the two different areas are areas of these two surfaces. When defined intrinsically in terms of quantum geometry described by  $\tilde{E}^a$  operator (as a collection of the 2-flats orthogonal to  $X_p^i$ ), the surface  $S$  is fluctuating due to quantum mechanical uncertainties, as we sketched above. Without a vector  $r^i$  this is the only surface we have, and its area is given by (1). However, if one chooses a vector  $r^i$  and defines  $S$  as being spanned by the 2-flats orthogonal to  $r^i$ , the surface is not fluctuating, and its area is given by (2). The two areas coincide in the limit of large spins (relative small fluctuations).

Concluding, we would like to repeat that in our opinion both area operators studied in this note are physically interesting. It is the context of a problem that should decide which operator should be used to measure the area of a surface. Here we gave a heuristic interpretation of the two areas as those of fluctuating and non-fluctuating surfaces. This interpretation, however, should be considered as only suggestive. More work is necessary to make it precise.

The last comment is that, in our opinion, it is the area (2) that should be used to measure the area of black hole horizon in the approach of [6]. In this approach the black hole horizon is treated as an internal spacetime boundary, where a special set of boundary conditions is imposed. One of this boundary conditions is exactly the requirement that there is a fixed unit vector  $r^i$  on the spatial cross-section of the horizon. Moreover, the boundary conditions also fix several components of the triad field  $\tilde{E}^a$  on the horizon: 1,2-components in the coordinate system used in (3). These components of the triad field are not subject to quantization, and can thus be used to identify the internal space at each point of  $S$  with the tangent space, as above. With this identification the surface  $S$  can be defined as spanned by the 2-flats orthogonal to  $r^i$ . As we sketched above, the boundary surface defined with respect to  $r^i$  is not fluctuating. Thus, one has to use the formula (2) for the area eigenvalues. The usage of the area spectrum (2) instead of (1) resolves the following puzzle in the approach [6] noted by Carlip [7]. The quantity  $A/8\pi\ell_P^2\gamma$ , where  $A$  is the horizon area, plays the role

of the level of the associated Chern-Simons theory (see [6] for details). However, the level is required to be an integer, which is not the case if the area spectrum is given by (1). If, on the other hand, one uses the spectrum (2), which we argued is the right spectrum to be used in the context of black holes, the level is given by the sum of spins labelling the black hole state, which is an integer due to the requirement of gauge invariance.

The following important point should be made regarding our proposal to use the area spectrum (2) as relevant in the context of black holes. The spectrum (2) is equidistant, i.e., the separation between two adjacent eigenvalues is constant, and equal to  $4\pi\ell_P^2\gamma$ . Thus, this area spectrum effectively reduces to the one proposed in 1974 by Bekenstein [8]. However, as it was recently shown by Bekenstein and Mukhanov [9], such area spectrum leads to the black hole emission spectrum qualitatively different from the one predicted by the semi-classical calculation of Hawking [10]. More precisely, the spectrum proposed by Bekenstein leads to the purely discrete emission spectrum, consisting of distinct, equidistant emission lines, while the semi-classical spectrum is continuous. However, as it was shown first in [11], the spectrum (1) is *not* equidistant, the separation between two adjacent lines for large horizon areas  $A$  goes as  $1/\sqrt{A}$ . Thus, if the black hole is described by (1), although the area spectrum is discrete, the separation between eigenvalues is very small for large areas, and the emission spectrum effectively reproduces continuous spectrum. This may lead one to conclude that (1), not (2), is the correct spectrum to describe the quantum black hole. However, we have argued that the difference between the two spectra (1), (2) is due to quantum fluctuations of the surface whose area is being measured, - in this case the quantum event horizon. This suggests that, in the case the spectrum (2) is used, the distinct lines in this spectrum get ‘smeared’ out by quantum fluctuations. Thus, the spectrum (2) will lead to qualitatively the same black hole emission spectrum as (1). The picture is similar to the one considered by Mäkelä [12]. Thus, the usage of the spectrum (2) *does not* seem to lead to any undesirable properties of the black hole emission spectrum.

This can be made more precise using the analysis of the black hole emission spectrum given in [13]. Considering quantum transitions in which the spin labelling one of the edges changes, one obtains the spectrum consisting of distinct emission lines, with the enveloping curve being thermal. One finds that the separation between lines in the emission spectrum is  $\Delta(\hbar\omega/2\pi T) = \Delta(j) = 1/2$ , where  $\omega$  is the frequency of the quantum emitted and  $T$  is the black hole temperature. As we have discussed above, the contribution to the horizon area from each edge labelled with spin  $j$  should be thought of as fluctuating, with the amplitude of fluctuations being  $\sqrt{j(j+1)} - j$ . In the black hole transition processes the main role is played by transitions with the final state being  $j = 1/2$ . Thus, taking  $j = 1/2$ , we get for the fluctuation amplitude  $\sqrt{3}/2 - (1/2) \approx 0.37$ . The width  $D$  of the emission line is proportional to twice the fluctuation amplitude  $D(\hbar\omega/2\pi T) \approx 2 * 0.37$ , which is larger than the separation  $\Delta(\hbar\omega/2\pi T)$  between the lines. Thus, the emission spectrum is effectively continuous.

The above argument is tentative. A more detailed study is necessary to understand all implications of the presence of the two area spectra in the theory. Note that an additional motivation for the usage of spectrum (2) comes from the study of rotating black holes, in particular extremal ones. As it is shown in [14], it is spectrum (2) that is compatible with certain desirable properties of the angular momentum in the theory.

Another implication of our results in the context of black holes is that the numerical value of the proportionality coefficient between the black hole entropy and the area changes. In the approach [6], the black hole entropy turns out to be given by  $N \ln 2$ , where  $N$  is the number of transverse intersections of spin 1/2 that is needed to give the total horizon area of black hole  $A$ . Using the area spectrum (2) instead of (1), we get  $N = A/4\pi\ell_P^2\gamma$ . This implies that the entropy is given by

$$S = \frac{\gamma_0}{\gamma} \frac{A}{4\ell_P^2}, \quad \gamma_0 := \frac{\ln 2}{\pi} \approx 0.22. \quad (10)$$

This is similar in the form to the result obtained in [6], the only difference being the numerical value of coefficient  $\gamma_0$ . The entropy still cannot be compared to Bekenstein-Hawking formula because of the presence of the free parameter  $\gamma$  in (10).

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## APPENDIX A: CONVENTIONS

The phase space of the theory is that of SU(2) Yang-Mills: the configurational variable is described by an SU(2) connection  $A_a$ , and its conjugated momentum is  $\tilde{E}^a$ , which can be thought of as a Lie algebra valued field. The momentum  $\tilde{E}^a$  has the geometrical interpretation of an orthonormal triad field with density weight one

$$\tilde{g}^{ab} = -\text{Tr}(\tilde{E}^a \tilde{E}^b). \quad (A1)$$

All traces in this paper are traces in the fundamental representation of SU(2). The density weight of  $\tilde{E}$  is indicated by the single ‘tilde’ over the symbol. It is convenient to introduce the components of fields  $A_a, \tilde{E}^a$  with respect to a basis in the Lie algebra. We take

$$A_a := -\frac{i}{2} \tau^i A_a^i \quad (A2)$$

$$\tilde{E}^a := -\frac{i}{\sqrt{2}} \tau^i \tilde{E}^{ai},$$

which are the standard conventions in the literature. Here  $\tau^i$  are the usual Pauli matrices ( $\tau^i \tau^j = i\epsilon^{ijk} \tau^k + \delta^{ij}$ ). The Poisson brackets between the canonical fields are given by

$$\{A_{ai}(x), \tilde{E}^{bj}(y)\} = 8\pi G \gamma \delta_a^b \delta_i^j \tilde{\delta}^3(x, y), \quad (A3)$$

where  $\delta_a^b, \delta_i^j$  are Kronecker deltas, and  $\tilde{\delta}^3$  is Dirac’s delta-function (densitized) in three dimensions. Note that the connection variable we use is the real connection  $A = \Gamma_a - \gamma K$ , where  $\gamma$  is the Immirzi parameter and  $\Gamma, K$  are the spin connection and the extrinsic curvature correspondingly, while  $\tilde{E}^a$  is the usual densitized triad field. This explains the appearance of  $\gamma$  in the Poisson brackets (A3).

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